解答

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1.

$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}, \tag{1}$$

$$\Pi_+^Z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi_-^Z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2}$$

$$\mathrm{Tr}\{\Pi_+^Z\rho\} = |\alpha|^2, \quad \mathrm{Tr}\{\Pi_-^Z\rho\} = |\beta|^2 \tag{3}$$

2.

$$\Pi_+^X = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \Pi_-^X = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{4}$$

$$\begin{aligned} \mathrm{Tr}\{\Pi_+^X\rho\} &= \mathrm{Tr}\left[\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}\right] \\ &= \frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} |\alpha|^2+\alpha^*\beta & \alpha\beta^*+|\beta|^2 \\ |\alpha|^2+\alpha^*\beta & \alpha\beta^*+|\beta|^2 \end{pmatrix}\right] \\ &= \frac{1}{2}(|\alpha|^2+\alpha^*\beta+\alpha\beta^*+|\beta|^2) \\ &= \frac{1}{2}+\mathrm{Re}[\alpha^*\beta], \end{aligned} \tag{5}$$

$$\begin{aligned} \mathrm{Tr}\{\Pi_-^X\rho\} &= \mathrm{Tr}\left[\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}\right] \\ &= \frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} |\alpha|^2-\alpha^*\beta & \alpha\beta^*-|\beta|^2 \\ -|\alpha|^2+\alpha^*\beta & -\alpha\beta^*+|\beta|^2 \end{pmatrix}\right] \\ &= \frac{1}{2}(|\alpha|^2+\alpha^*\beta-\alpha\beta^*+|\beta|^2) \\ &= \frac{1}{2}-\mathrm{Re}[\alpha^*\beta], \end{aligned} \tag{6}$$

3.

$$\Pi_+^Y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad \Pi_-^Y = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \tag{7}$$

$$\begin{aligned} \mathrm{Tr}\{\Pi_+^Y\rho\} &= \mathrm{Tr}\left[\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}\right] \\ &= \frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} |\alpha|^2-i\alpha^*\beta & \alpha\beta^*-i|\beta|^2 \\ i|\alpha|^2+\alpha^*\beta & i\alpha\beta^*+|\beta|^2 \end{pmatrix}\right] \\ &= \frac{1}{2}(|\alpha|^2-i\alpha^*\beta+i\alpha\beta^*+|\beta|^2) \\ &= \frac{1}{2}+\mathrm{Im}[\alpha^*\beta], \end{aligned} \tag{8}$$

$$
\begin{aligned}
\mathrm{Tr}\{\Pi_-^Y\rho\} &= \mathrm{Tr}\left[\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}\right] \\
&= \frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} |\alpha|^2 + i\alpha^*\beta & \alpha\beta^* + i|\beta|^2 \\ -i|\alpha|^2 + \alpha^*\beta & -i\alpha\beta^* + |\beta|^2 \end{pmatrix}\right] \\
&= \frac{1}{2}(|\alpha|^2 + i\alpha^*\beta + (-i)\alpha\beta^* + |\beta|^2) \\
&= \frac{1}{2} - \mathrm{Im}[\alpha^*\beta], \qquad (9)
\end{aligned}
$$

4. $\mathrm{Tr}\{\Pi\rho\}$ は、状態 $|\psi\rangle$ を観測して $|\phi\rangle$ が得られる確率である。